Kuwait University | MATH 102 | March 22, 2001

Math. Dept. | **First Midterm** | Duration 75 minutes

Answer the following questions. **Calculators and mobile telephones are not allowed.**

1. Show that: (5 points each)

(a) $\tan(\sin^{-1} x) - \cot(\cos^{-1} x) = 0$, for $x \in (0, 1)$.

(b) $\log_{x^2} x + \log_x x^2 = \frac{5}{2}$, for $x > 0, x \neq 1$.

2. Find $\frac{dy}{dx}$, if (5 points each)

(a) $y = (\log_5(\sqrt{x} + 1))^{\tanh x}$.

(b) $\sin^{-1}(x + y) + y \tan^{-1} x + \sinh(xy) = 0$.

3. Evaluate the following integrals: (5 points each)

(a) $\displaystyle\int \frac{\sec(2^{-x})}{2^x} dx$

(b) $\displaystyle\int \frac{dx}{x\sqrt{x^8 - 4}}$

(c) $\displaystyle\int \frac{3 - 2\tan x}{2 + 3\tan x} dx$

4. Find the following limits: (5 points each)

(a) $\displaystyle\lim_{x \to 1} \frac{\ln(ex) - e^{x-1}}{(x-1)^2}$.

(b) $\displaystyle\lim_{x \to 0^+} (1 + \sin 2x)^{\csc 3x}$.

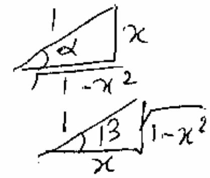
5. Let $f(x) = \ln \frac{1 + e^x}{2 + e^x}$, $x \in (-\infty, \infty)$.

Find f^{-1} and state the domain of f^{-1}. (5 points)

1(a) $\tan(\sin^{-1}x) - \cot(\cos^{-1}x) = 0$

L.H.S let $\sin^{-1}x = \alpha \Rightarrow \sin\alpha = x$

$\cos^{-1}x = \beta \Rightarrow \cos\beta = x$



Now L.H.S $\tan\alpha - \cot\beta$

$= \frac{x}{\sqrt{1-x^2}} - \frac{x}{\sqrt{1-x^2}} = 0$ R.H.S

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b) $\log_{x^2} x + \log_x x^2 = 5/2$

L.H.S let $\log_{x^2} x = a \quad \therefore x = (x^2)^a \Rightarrow x^{2a}$

$\therefore \quad x = x^{2a} \Rightarrow 2a = 1 \Rightarrow a = \frac{1}{2}$

Let $\log_x x^2 = b \Rightarrow x^2 = x^b \Rightarrow b = 2.$

Hence L.H.S $= a + b$

$= \frac{1}{2} + 2 = 5/2$ R.H.S

Q2 (a) $y = \left(\log_5(\sqrt{x}+1)\right)^{\tanh x}$.

$\ln y = \tanh x \cdot \ln\left[\log_5(\sqrt{x}+1)\right]$.

$\therefore \frac{1}{y} y' = \tanh x \cdot \frac{1}{(\sqrt{x}+1)\ln 5} \cdot \frac{1}{2} x^{-1/2} + \log_5(\sqrt{x}+1) \cdot \operatorname{sech}^2 x$.

$\therefore y' = \left(\log_5(\sqrt{x}+1)\right)^{\tanh x}\left[\frac{\tanh x}{2\sqrt{x}\,(\sqrt{x}+1)\ln 5} + \operatorname{sech}^2 x \log_5(\sqrt{x}+1)\right]$

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b) $\sin^{-1}(x+y) + y\tan^{-1}x + \sinh(xy) = 0$

$\frac{1}{\sqrt{1-(x+y)^2}}\left(1+\frac{dy}{dx}\right) + \frac{y}{1+x^2} + (\tan^{-1}x)y' + \cosh(xy)\left[xy'+y\right] = 0$

$\frac{1}{\sqrt{1-(x+y)^2}} + \frac{1}{\sqrt{1-(x+y)^2}}y' + \frac{y}{1+x^2} + (\tan^{-1}x)y' + (x\cosh xy)\,y' + y\cosh xy = 0$

$$\left(\frac{1}{\sqrt{1-(x+y)^2}} + \tan^{-1} x + x\cosh xy\right) y' = -\left(\frac{1}{\sqrt{1-(x+y)^2}} + \frac{y}{1+x^2} \ldots\right)$$

$$\therefore \quad y' = -\frac{\left[\frac{1}{\sqrt{1-(x+y)^2}} + \frac{y}{1+x^2} + y\cosh xy\right]}{\left[\frac{1}{\sqrt{1-(x+y)^2}} + \tan^{-1} x + x\cosh xy\right]}$$

Q3 a) $\int \frac{\sec(2^{-x})}{2^x}\,dx$ let $2^{-x} = u$

$-2^{-x}\ln 2\,dx = du$

$\therefore \quad 2^{-x}\,dx = -\frac{1}{\ln 2}\,du$

$\therefore -\frac{1}{\ln 2}\int \sec u\,du = -\frac{1}{\ln 2}\ln|\sec u + \tan u| + c$

$= -\frac{1}{\ln 2}\ln\left[\sec 2^{-x} + \tan 2^{-x}\right] + c$

b) $\int \frac{1}{x\sqrt{x^8-4}}\,dx = \int \frac{x^3}{x^4\sqrt{x^8-4}}\,dx$

Let $x^4 = u \Rightarrow 4x^3\,dx = du$

$\therefore \quad x^3\,dx = \frac{1}{4}\,du$

$\therefore \frac{1}{4}\int \frac{1}{u\sqrt{u^2-2^2}}\,du = \frac{1}{4}\cdot\frac{1}{2}\sec^{-1}\frac{u}{2} + c$

$\therefore \quad \frac{1}{8}\sec^{-1}\frac{x^4}{2} + c$

c) $\int \frac{3-2\tan x}{2+3\tan x}\,dx = \int \frac{3\cos x - 2\sin x}{2\cos x + 3\sin x}\,dx$

let $2\cos x + 3\sin x = u$

$(3\cos x - 2\sin x)\,dx = du$

$= \int \frac{1}{u}\,du = \ln u + c$

$= \ln|2\cos x + 3\sin x| + c$

Q.4 (a) $\lim_{x\to 1} \frac{\ln(ex) - e^{x-1}}{(x-1)^3}$ $\frac{0}{0}$

$\therefore \lim_{x\to 1} \frac{\frac{1}{x} - e^{x-1}}{3(x-1)^2}$ $\frac{0}{0}$.

$\lim_{x\to 1} \frac{-\frac{1}{x^2} - e^{x-1}}{6(x-1)} = \frac{-2}{0} = -\infty$.

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b) $\lim_{x\to 0^+} (1+\sin 2x)^{\csc 3x}$ 1^{∞}

$y = (1+\sin 2x)^{\csc 3x}$

$\therefore \ln y = \csc 3x \ln(1+\sin 2x)$

$\ln y = \lim_{x\to 0^+} \csc 3x \ln(1+\sin 2x)$ $0\cdot\infty$

$= \lim_{x\to 0^+} \frac{\ln(1+\sin 2x)}{\sin 3x}$ $\frac{0}{0}$

$\therefore \lim_{x\to 0} \frac{\frac{2\cos 2x}{1+\sin 2x}}{3\cos 3x} = \frac{\cancel{\phi}2}{3}$.

$\therefore y = e^{2/3}$ Ans

Q5 $f(x) = \ln \frac{1+e^x}{2+e^x}$ $x \in (-\infty, \infty)$

Range is $(\frac{1}{2}, 1)$

$(\ln(\frac{1}{2}), \ln(1))$

$(\ln(\frac{1}{2}), 0)$.

$y = \ln \frac{1+e^x}{2+e^x}$

$e^y = \frac{1+e^x}{2+e^x}$.

$2e^y + e^y \cdot e^x = 1 + e^x$.

$2e^y - 1 = (1-e^y)e^x$

$e^x = \frac{2e^y - 1}{1-e^y}$.

$x = \ln\left|\frac{2e^y - 1}{1-e^y}\right|$

$f^{-1}(x) = \ln\left|\frac{2e^x - 1}{1-e^x}\right|$